Endpoint estimates, Extrapolation Theory and the Bochner-Riesz operator

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May 23, 2015

1. Extrapolation theory
   - Classical A_p theory
   - $\widehat{A}_p$ weights and a new extrapolation
   - $(\varepsilon, \delta)-$ atomic operators

2. The Bochner-Riesz operator at the critical index
   - A restricted weak-type estimate
   - Averages of operators

The Hardy-Littlewood maximal operator

Definition

We consider the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

L^p and $L^{p,\infty}$ spaces

Definition

Given a weight $w > 0$*, for* $1 \leq p < \infty$*:*

$$\|f\|_{L^p(w)}^p = \int_{\mathbb{R}^n} |f(x)|^p w(x) dx,$$

and also

$$\|f\|_{L^{p,\infty}(w)}^p = \sup_{t>0} t^p \int_{\{|f|>t\}} w(x) dx.$$

It holds that $L^p \subsetneq L^{p,\infty}$.

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$$\|Tf\|_{L^p} \lesssim \|f\|_{L^p} \quad \rightsquigarrow \textbf{STRONG-TYPE } (p,p)$$

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$$\|Tf\|_{L^{p,\infty}} \lesssim \|f\|_{L^p} \quad \rightsquigarrow \textbf{WEAK-TYPE } (p,p)$$

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It holds that $L^p \subsetneq L^{p,\infty}$. For an operator T, we will say

$\|T\chi_E\|_{L^{p,\infty}} \lesssim \|\chi_E\|_{L^p}$ $\rightsquigarrow$ **RESTRICTED WEAK-TYPE** (p,p)

A_p weights – Muckenhoupt (1972)

For every $1 < p < \infty$:

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For $p = 1$,

$$\|Mf\|_{L^{1,\infty}(u)} \lesssim \|f\|_{L^1(u)} \Leftrightarrow u \in A_1,$$

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We write, for $1 \leq p < \infty$, $\|Mf\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(w)} \Leftrightarrow w \in A_p$.

Characterization of A_p

P. Jones' Factorization:

$$w \in A_p \Leftrightarrow w = v^{1-p}u, \quad \text{for some } u, v \in A_1.$$

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Coifman - Rochberg's construction of A_1 weights:

$$v \in A_1 \Leftrightarrow v \approx (Mf)^\delta, \quad \text{for some } f \in L^1_{loc} \text{ and } 0 \leq \delta < 1.$$

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Therefore, we can think of A_p weights as those of the form:

Proposition

$$A_p = \left\{ (Mf)^{\delta(1-p)}u \,:\, f \in L^1_{loc}, 0 < \delta < 1 \text{ and } u \in A_1 \right\}.$$

An important property: Reverse Hölder

With this characterization, for every $w \in A_p$:

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So it is immediate to see that, for some small ε, $\left(\varepsilon < \min\left\{\frac{1-\delta}{\delta}, \frac{1-\beta}{\beta}\right\}\right)$,

$$w^{1+\varepsilon} = (Mf)^{\delta'(1-p)} (Mg)^{\beta'},$$

with $0 < \delta', \beta' < 1$ and hence,

$$w^{1+\varepsilon} \in A_p.$$

Rubio de Francia

In this setting:

Theorem (Rubio de Francia's Extrapolation - 1984)

Given a sublinear operator T such that for some $1 \leq p_0 < \infty$ we have

$$\|Tf\|_{L^{p_0,\infty}(w)} \lesssim \|f\|_{L^{p_0}(w)} \qquad \text{for every } w \in A_{p_0},$$

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Remark

*Notice that the endpoint $\mathrm{p} = 1$ **cannot** be reached.*

Goal

Remark (This is the plan...)

$$\|Tf\|_{L^{p_0,\infty}(w)} \lesssim \|f\|_{L^{p_0}(w)}, \quad \forall w \in A_{p_0}$$

$$\not\Downarrow$$

$$\|Tf\|_{L^{1,\infty}(u)} \lesssim \|f\|_{L^1(u)}, \quad \forall u \in A_1.$$

Take for instance $T = M \circ M$.

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A **weaker** assumption on the **boundedness**.

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We only get restricted weak-type (1,1), but we will usually deal with it.

But, how do we find these new weights $\widehat{A}_p$??

Searching the weights

Kerman and Torchinsky (1982): For $1 \leq p < \infty$:

$$\|M\chi_E\|_{L^{p,\infty}(w)} \lesssim \|\chi_E\|_{L^p(w)} \Leftrightarrow w \in A_p^{\mathcal{R}}$$

where, for $1 \leq p < \infty$, $w \in A_p^{\mathcal{R}}$ if

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Remark

$A_1^{\mathcal{R}} = A_1$... *we'll see why this makes sense!!*

Searching the weights

The key fact for the new extrapolation:

Theorem (Carro, Grafakos, Soria)

Given a locally integrable function f and $u \in A_1$, then

$$(Mf)^{1-p} u \in A_p^{\mathcal{R}},$$

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Definition

$$\widehat{A}_p = \left\{w = (Mf)^{1-p}u, \quad \textit{where } f \in L^1_{loc}, u \in A_1\right\} \subseteq A_p^{\mathcal{R}},$$

The extrapolation result

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Remark

*Here we **reach** the endpoint $\mathbf{p} = \mathbf{1}$, and $\widehat{A}_p = A_1$!*

Overview

$$A_p = \left\{(Mf)^{\delta(1-p)}u \,:\, \delta < 1, u \in A_1\right\}$$

T w.t (p_0, p_0) for every $w \in A_{p_0}$

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T s.t (p, p) for every $w \in A_p$

$$(1 \le p_0 < \infty, \quad 1 < p < \infty)$$

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A weaker hypothesis

If for some $1 < p_0 < \infty$

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But we also have: If for some $1 < p_0 < \infty$

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then, for every $1 \leq p < \infty$

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Statement

Theorem (Carro, D-S)

Given an operator T such that for some $1 \leq p_0 < \infty$ and every $u \in A_1$:

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Theorem (v. 2.0)

Given an operator T such that for every $u \in A_1$ there is $1 \le p_0 < \infty$ such that:

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From restricted weak-type to weak-type

Question

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In general, it is not true!! For instance, take the operator

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which is related to Bourgain's return time theorems.

It is immediate to check that

$$A\chi_E \leq M\chi_E,$$

so it is of restricted weak-type (1,1) for weights in A_1. However it is not of weak-type (1,1)!!

From restricted weak-type to weak-type

Definition

A sublinear operator T *is* $(\varepsilon, \delta)-$*atomic if, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *s.t.*

$$\|Ta\|_{L^1+L^\infty} \leq \varepsilon \|a\|_1,$$

for every δ*-atom* a $\left(\int a = 0 \text{ and } \operatorname{supp} a \subseteq Q \text{ with } |Q| \leq \delta\right)$.

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For instance:

- $$Tf(x) = K * f(x),$$

  with $K \in L^p$ for some $1 \leq p < \infty$, is $(\varepsilon, \delta)-$atomic.

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For instance:

- $$Tf(x) = K * f(x),$$
  with $K \in L^p$ for some $1 \leq p < \infty$, is $(\varepsilon, \delta)-$atomic.
- If $\{T_n\}_n$ is a sequence of $(\varepsilon, \delta)-$atomic operators, then:
  $$T^*f(x) = \sup_n |T_n f(x)|, \quad \text{and} \quad Tf(x) = \left(\sum_n |T_n f(x)|^q \right)^{1/q},$$
  are $(\varepsilon, \delta)-$atomic approximable, for every $q \geq 1$.

From restricted weak-type to weak-type

Proposition

If T is $(\varepsilon, \delta)-$ atomic (approximable), then for every $u \in A_1$:

Restricted Weak-Type (1,1) $\iff$ *Weak-Type (1,1).*

Remark

This explains why $A_1^{\mathcal{R}} = A_1$!!

Applications

More examples:

(i) If $u(x,t) = P_t * f(x)$ is the Poisson integral of f, the Lusin area integral is defined by

$$S_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$.

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(ii) The Littlewood-Paley g-function

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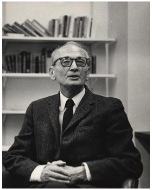
(iii) The intrinsic square function G_α (introduced by M. Wilson), Haar shift operators, singular integrals, averages of operators...

Bochner-Riesz

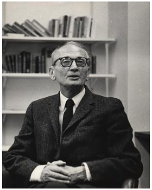


Definition (The Bochner-Riesz operator)

Given $\lambda > 0$,

$$\widehat{(T_\lambda f)}(\xi) = (1 - |\xi|^2)_+^\lambda \widehat{f}(\xi).$$

Bochner-Riesz



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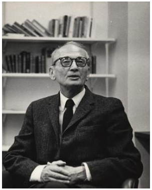
When $\lambda > \frac{n-1}{2}$,

$$|T_\lambda f| \lesssim Mf.$$

Bochner-Riesz



Definition (The Bochner-Riesz operator)

Given $\lambda > 0$,

$$\widehat{(T_\lambda f)}(\xi) = (1 - |\xi|^2)_+^\lambda \widehat{f}(\xi).$$

When $\lambda > \frac{n-1}{2}$,

$$|T_\lambda f| \lesssim Mf.$$

From now on, we fix $\lambda = \frac{n-1}{2} \rightsquigarrow$ **THE CRITICAL INDEX**.

The (short) story

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- In 1992, X. Shi and Q. Sun prove that T_λ is of strong-type (p, p) for A_p.
- In 1996, A. Vargas obtains the weak-type (1,1) for weights in A_1.

Our result

We prove that

Theorem (Carro, D-S)

Given $u \in A_1$, *for some* $1 < p_0 < \infty$

$$\|T_\lambda(\chi_E)\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim u(E)^{1/p_0}.$$

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- This is stronger than A. Vargas' result about the weak-type (1,1) for A_1 weights.
- It also allows to get endpoint results for average operators, as we will see later on.

Decomposition of the kernel

We use the standard decomposition of the convolution kernel:

$$T_\lambda f = K * f = \left(\sum_{j=1}^{\infty} K_j \right) * f,$$

with $|K_j(x)| \lesssim 2^{-nj} \chi_{B(0,2^j)}(x)$. Clearly, for every $j \geq 1$,

$$|K_j * f(x)| \lesssim Mf(x).$$

Reverse Hölder to the rescue in A_p–theory

Fix $w \in A_2$. We have, for every $j \geq 1$:

- $\|K_j * f\|_2 \lesssim 2^{-c_n j} \|f\|_2,$ (M. Christ)

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We take $w^{1+\varepsilon} \in A_2$, $\theta = \frac{1}{1+\varepsilon}$, and we are done for $p = 2$. We use Rubio de Francia's extrapolation to conclude that

$$T_\lambda : L^p(w) \longrightarrow L^p(w), \quad w \in A_p \quad (1 < p < \infty)$$

Idea Behind the Proof of:

Theorem (Carro, D-S)

Given $u \in A_1$*, for some* $1 < p_0 < \infty$

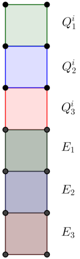
$$\|T_\lambda(\chi_E)\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim u(E)^{1/p_0}.$$

- ✓ Decomposition of the kernel $K = \sum_j K_j$,
- ✓ Decomposition of the set $E = \bigcup_k E_k$,
- ✓ Main Lemma.

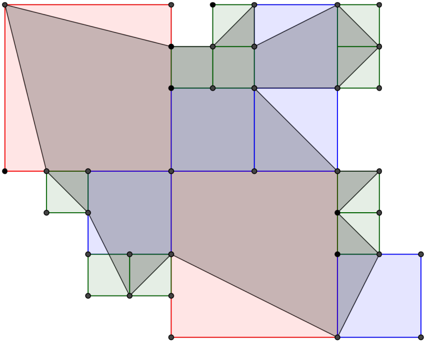
Decomposition of E

Given $0 < \alpha < 1$, we have disjoint dyadic cubes $\{Q_i^k\}_{i,k}$ and we can decompose a set $E \subseteq \mathbb{R}^n$

$$E = \bigcup_{k \geq 0} E_k = \bigcup_{k \geq 0} E \cap \left(\cup_i Q_i^k \right), \quad \text{with } \frac{|E \cap Q_i^k|}{2^{nk}} \approx \alpha.$$



$E = E_1 \cup E_2 \cup E_3$



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Lemma

Let $0 < \alpha < 1$, $E = \cup_k E_k$ *and* $u \in A_1$. *Then, for every* $1 \leq s < \infty$, *if*

$$F_s(x) := \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}(x),$$

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(e) $\|F_s\|_{L^2((M\chi_E)^{-\theta} u)}^2 \lesssim 2^{-s\beta} \alpha^{1-\theta} u(E)$.

Weighted results for T_λ

So we have that

$$\|T_\lambda(\chi_E)\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim u(E)^{1/p_0}, \quad (u \in A_1, p_0 > 1),$$

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$$\|T_\lambda f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad (p > 1, w \in A_p).$$

An application

Consider a radial Fourier multiplier

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

where $m \in L^\infty(0,\infty)$ such that $t^{\frac{n-1}{2}} D^{\frac{n+1}{2}} m(t) \in L^1(0,\infty)$.

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where $m \in L^\infty(0,\infty)$ such that $t^{\frac{n-1}{2}} D^{\frac{n+1}{2}} m(t) \in L^1(0,\infty)$. Then, one can prove that there is $\Phi \in L^1(0,\infty)$ such that

$$m(|\xi|^2) = \int_0^\infty \left(1 - \frac{|\xi|^2}{s^2}\right)_+^{\frac{n-1}{2}} \Phi(s)ds.$$

An application

With this, we have

$$T_m f(x) = \int_0^\infty B_s f(x)\Phi(s)ds,$$

where

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If K is the kernel associated with T_λ, and K_s with B_s, then

$$K_s(x) = s^n K(sx).$$

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From the estimate for T_λ, we deduce the uniform bound

$$\|B_s \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim u(E)^{1/p_0}.$$

An application

Using now that $L^{p_0,\infty}$ is a Banach space, we can use Minkowski's inequality!!

$$\|T_m\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \int_0^\infty \|B_s\chi_E\||\Phi(s)|ds \lesssim u(E)^{1/p_0}.$$

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From this, we extrapolate down to $p = 1$:

$$\|T_m f\|_{L^{1,\infty}(u)} \lesssim \|f\|_{L^1(u)}, \quad u \in A_1.$$

Remark

Notice that if we only have a weak-type (1,1) estimate, averages do not inherit this property.

Muchas Gracias!